# ON CANONICAL THREEFOLDS NEAR THE NOETHER LINE

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Let $X$ be a canonical $n$-fold, that is a complex irreducible projective variety of dimension $n$ whose singularities are at worst canonical and whose canonical class $K_X$ is ample. If $n = 2$ we will use the word "surface" instead of the word 2-fold.

As usual we denote by $p_g := h^0(X, K_X)$ its geometric genus and by $K_X^n$ its canonical volume. The volume of a canonical surface is an integer respecting the M. Noether inequality $K^2 \geq 2p_g - 4$.

The moduli space of the canonical surfaces $X$ with $K_X^2 = 2p_g(X) - 4$ has been described by Horikawa. According to Horikawa's description, if $K_X^2 = 2p_g(X) - 4$ and $p_g \geq 7$, then there is a fibration $f \colon X \to \mathbb{P}^1$ with fibres of genus 2. Let $f \colon X \to B$ be a genus 2 fibration, that is a surjective morphism onto a smooth projective curve $B$ whose general fibre is a curve of genus 2. The canonical ring of a fibre $F$, the ring $\bigoplus_d H^0(F, dK_F)$, is of one of the following two forms:

| 2-*connected fibres* | | 2-*disconnected fibres* |
|---|---|---|
| $\dfrac{\mathbb{C}[x_0, x_1, z]}{f_6(x_0, x_1, z)}$ | or | $\dfrac{\mathbb{C}[x_0, x_1, y, z]}{f_2(x_0, x_1), f_6(x_0, x_1, y, z)}$ |

with $\deg x_j = 1$, $\deg y = 2$, $\deg z = 3$, $\deg f_d = d$.

Moreover (see [CP06])

$$K_X^2 = 2p_g\,(X) - 4 + 6b - 2h^1(X, \mathcal{O}_X) + \deg \tau$$

where $b$ is the genus of the base curve $B$, and $\tau$ is an effective divisor on $B$ supported on the image of the 2-disconnected fibres. By an inequality of Jongmans and Debarre, if $K_X^2 < 2p_g$, then $X$ is regular (that means $h^1(X, \mathcal{O}_X) = 0$), which in turn implies $b = 0$. In particular

$$K_X^2 = 2p_g\,(X) - 4 \Leftrightarrow h^1(X, \mathcal{O}_X) = 0 \text{ and } \tau = 0.$$

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We can then state the following.

**Definition 1.** A simple genus 2 fibration is a projective variety $X$ with a morphism $f$ on a smooth projective curve $B$ such that

(1) all singularities of $X$ are canonical;
(2) $K_X$ is $f$-ample;
(3) for *all* $p \in B$, the canonical ring of the fibre $X_p = f^{-1}p$ is of the form $\frac{\mathbb{C}[x_0,x_1,z]}{f_6(x_0,x_1,z)}$, with $\deg x_i = 1$, $\deg z = 3$, $\deg f_6 = 6$

and the mentioned results give the following

**Theorem 2.** *Let $X$ be a projective variety of dimension 2 with at worst canonical singularities and $p_g \geq 7$. Then $X$ is a regular simple genus 2 fibration if and only if $X$ is a canonical surface and $K_X^2 = 2p_g - 4$.*

If $n = 3$ [CCJ20b, CCJ20a] proved that the *Noether inequality for canonical 3-folds*

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$$

holds, unless (possibly) $5 \leq p_g \leq 10$. The first step of the proof is showing that, if $K_X^3$ is not much bigger than $\frac{4}{3}p_g(X) - \frac{10}{3}$, then there exists a fibration $f\colon X \dashrightarrow \mathbb{P}^1$ whose general fibre is a canonical surface with canonical volume 1 and geometric genus 2. These are classically known to be the hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$ with at worst canonical singularities.

This, together with Theorem 2, inspired the following definition

**Definition 3.** [CP23] A simple fibration in $(1,2)$-surfaces is a projective variety with a morphism $f$ on a smooth projective curve $B$ such that

(1) all singularities of $X$ are canonical;
(2) $K_X$ is $f$-ample;
(3) for *all* $p \in B$, the canonical ring of the fibre $X_p = f^{-1}p$ is of the form $\frac{\mathbb{C}[x_0,x_1,y,z]}{f_{10}(x_0,x_1,y,z)}$, with $\deg x_i = 1$, $\deg y = 2$, $\deg z = 5$, $\deg f_{10} = 10$.

In [CHPZ24] we generalize Theorem 2 to dimension 3 as follows.

**Theorem 4.** *Let $X$ be a projective variety of dimension 3 with at worst canonical singularities and $p_g \geq 23$. Then $X$ is a Gorenstein regular simple fibration in $(1,2)$-surfaces if and only if $X$ is a canonical 3-fold and $K_X^3 = \frac{4}{3}p_g - \frac{10}{3}$.*

One implication was proven in [CP23] where the other implication was conjectured. The inequality $p_g \geq 23$ is sharp, since there are simple fibrations in $(1,2)$-surfaces with $p_g = 13, 16, 19, 22$ and $K_X$ not ample: the map from these threefolds onto their canonical model is a small contraction, contracting a rational curve contained in the singular locus of $X$.

This allowed a complete classification of the moduli spaces $\mathcal{M}_{p_g}$ of the canonical threefolds with $p_g \geq 11$ and $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$. It was already known that $\mathcal{M}_{p_g}$ is empty unless $p_g + 2$ is divisible by 3. In [CHPZ24] we prove

**Theorem 5.** *If $p_g \geq 11$ and $p_g+2$ is divisible by 3 then $\mathcal{M}_{p_g}$ decomposes as union of $\left\lfloor \frac{p_g+6}{4} \right\rfloor$ unirational strata. The number of irreducible components of $\mathcal{M}_{p_g}$ is at most $\left\lfloor \frac{p_g+6}{4} \right\rfloor$ and at least $\left\lfloor \frac{p_g+6}{4} \right\rfloor - \left\lfloor \frac{p_g+8}{78} \right\rfloor$.*

It is worth mentioning that, for all $p_g$ only one or two strata contain smooth 3-folds. The remaining strata provide then examples of 3-folds with smoothable singularities that are not globally smoothable.

The following results

**Theorem 6** ([CP23])**.** *Let $f\colon X \to B$ be a simple fibration in $(1,2)$-surfaces. Then $X$ is 2-Gorenstein. If $X$ is regular then $K_X^3 = \frac{4}{3}p_g - \frac{10}{3} + \frac{N}{6}$, $N \in \mathbb{N}$. Moreover $X$ is Gorenstein if and only if $N=0$.*

**Theorem 7** ([HZ24])**.** *If $X$ is a canonical 3-fold with $p_g(X) \geq 11$ and $K_X^3 \leq \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{2}{6}$ then $X$ is 2-Gorenstein and $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3} + \frac{N}{6}$ with $N = 0,1,2$. Moreover $X$ is Gorenstein if and only if $N=0$.*

lead to the natural

**Question:** *Find the maximal $\epsilon > 0$ (if it exists) such that if $p_g >> 1$ then all threefolds $X$ with $\mathrm{vol}(X) < \frac{4}{3}p_g(X) - \frac{10}{3} + \epsilon$ are regular simple fibrations in $(1,2)-$surfaces.*

Theorems 4 and 7 show $\epsilon \geq \frac{1}{6}$. Proving $\epsilon \geq \frac{2}{6}$ would lead to a complete classification of the moduli space of canonical threefolds *on the second Noether line.*

On the opposite side we can prove that $\epsilon \leq \frac{4}{6}$ by constructing ([CHPZ]), for each positive integer $p_g$ divisible by 3, canonical 3-folds with $K^3 = \frac{4}{3}p_g + \frac{10}{3} + \frac{4}{6}$ of Gorenstein index 3, which implies by Theorem 6 that they are not simple fibrations in $(1,2)$-surfaces.

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